

# Perturbation theory for velocity-dependent potentials

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Received: 24 May 2002 /

Published online: 3 December 2002 – © Società Italiana di Fisica / Springer-Verlag 2002

Communicated by V. Vento

**Abstract.** In the presence of a velocity-dependent Kisslinger potential, the partial-wave, time-independent Schrödinger equation with real boundary conditions is written as an equation for the probability density. The changes in the bound-state energy eigenvalues due to the addition of small perturbations in the local as well as the Kisslinger potentials are determined up to second order in the perturbation. These changes are determined purely in terms of the unperturbed probability density, the perturbing local potential, as well as the Kisslinger perturbing potential and its gradient. The dependence on the gradient of the Kisslinger potential stresses the importance of a diffuse edge in nuclei. Two explicit examples are presented to examine the validity of the perturbation formulas. The first assumes each of the local and velocity-dependent parts of the potential to be a finite square well. In the second example, the velocity-dependent potential takes the form of a harmonic oscillator. In both cases the energy eigenvalues are determined exactly and then by using perturbation theory. The agreement between the exact energy eigenvalues and those obtained by perturbation theory is very satisfactory.

**PACS.** 03.65.Ge Solutions of wave equations: bound states – 31.15.Md Perturbation theory

## 1 Introduction

In quantum mechanics there are relatively few physically interesting problems that may be solved exactly. Consequently, approximation methods are vitally important in many applications of the theory. In non-relativistic quantum mechanics, nearly all the approximation schemes are developed starting from the Schrödinger equation. For small perturbations in the local potential, the time-independent perturbation theory estimates corrections to the bound-state energies and the corresponding wave functions by exploiting the orthogonality of different wave functions [1]. However, a different approach is presented in [2], which recasts the partial-wave, time-independent Schrödinger equation with real boundary conditions as an equation for the probability density. Up to third order in the perturbation, the changes in the bound-state energies are expressed purely in terms of the unperturbed probability density and the perturbing potential corresponding to a local potential only. Another work [3] derived a form of perturbation theory where the shift in energy and the change in the wave function of a state are obtained without a need for information about the other states.

The Schrödinger equation with a non-constant coefficient by the second-order derivative is an interesting and useful model for describing many physical problems. Such

an equation may be used to describe the motion of a particle with a spatially varying effective mass moving in some potential [4, 5]. The effective-mass approximation is an important tool used for the determination of the electronic properties of semiconductors [6] and quantum dots [7]. Another important application of the Schrödinger equation with a non-constant term is in the field of nuclear physics. Kisslinger developed a velocity-dependent potential that was successful in describing the scattering of mesons off complex nuclei, which predicted the predominantly  $p$ -wave nature of the elementary pion-nucleon coherent scattering [8]. The potential is expressed as

$$\nabla \cdot (\rho \nabla \psi) = \rho \nabla^2 \psi + \nabla \rho \cdot \nabla \psi. \quad (1)$$

Generally  $\rho$  represents the spatially varying nuclear density. Consequently, the second term on the right is sensitive to the diffuse edge in nuclei. Further, the first term is proportional to the kinetic energy and combines with the kinetic-energy term in the Schrödinger equation. For a particle of mass  $m$  and energy  $E$  moving in a reduced potential  $U(r)$  the time-independent Schrödinger equation including the Kisslinger term may be expressed as

$$(1 - \rho)u''(r) - \left[ u'(r) - \frac{u(r)}{r} \right] \rho'(r) = [U(r) - E] u(r), \quad (2)$$

where  $u(r)$  is a reduced bound-state wave function,  $E$  and  $U(r)$  are measured in units of  $\hbar^2/2m$ . Finally, the

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prime denotes differentiation with respect to the radial variable  $r$ .

In sect. 2 we shall outline the boundary conditions that  $U(r)$  and  $\rho(r)$  must satisfy so that eq. (2) has physically acceptable solutions. The probability density equation that includes the non-constant term  $\rho(r)$  will be developed in sect. 3 and some of its properties will be discussed. In this paper, it is our aim to develop a time-independent perturbation theory when small, spherically symmetric perturbations in the local and velocity-dependent potentials are added. This will be done in sect. 4 in the framework of the probability density equation. It will also be shown that the expressions for the energy corrections are valid for an arbitrary orbital angular-momentum quantum number  $l$ . In sect. 5 we shall consider two explicit examples, square well and harmonic-oscillator potentials, to demonstrate the validity of the derived expressions.

## 2 Boundary conditions on the potentials

For eq. (2) to have physically acceptable solutions the potential terms must satisfy the following conditions [9,10]:

$$\int_0^\infty r|U(r)|dr < \infty, \quad \int_0^\infty r^2|U(r)|dr < \infty. \quad (3)$$

and

$$\int_0^\infty r|\rho''(r)|dr < \infty, \quad \int_0^\infty r^2|\rho(r)|dr < \infty. \quad (4)$$

The conditions in eq. (3) suggest that  $U(r)$  diverges slower than  $1/r$  for small  $r$  and decays faster than  $1/r^3$  at infinity. However, for the Kisslinger potential, the first equation in (4) demands that  $\rho'(r)$  diverges less than  $1/r$  close to the origin. This condition is satisfied if in the vicinity of the origin we have  $\rho(r) \approx b_0 r^p$  where  $p > 0$  and  $b_0$  is a constant. Considering eq. (2) it is clear that  $\rho(r)$  must be bounded away from 1. Hence, to satisfy the boundary condition on the velocity-dependent part of the potential we must have  $\rho(r) < 1$  for all  $r$ . The second condition in (4) implies that  $\rho(r)$  falls off faster than  $1/r^3$  for large distances. The above boundary conditions guarantee that the  $\lim_{r \rightarrow 0} u(r) = 0$ .

## 3 The probability density equation

For stationary problems where the reduced bound-state wave function,  $u(r)$ , is real one may work with the probability density  $P(r)$  defined as

$$P(r) = [u(r)]^2. \quad (5)$$

Using the above equation we may transform the Schrödinger equation in (2) into one for the probability density. Multiplying (2) by  $2u^3$  and adding the term

$2(1-\rho)u^2u'^2$  to both sides it is straightforward to show that  $P(r)$  satisfies the following differential equation:

$$(1-\rho) \left[ PP'' - \frac{P'^2}{2} \right] - 2P^2(U-E) - \rho'PP' + \frac{2\rho'P^2}{r} = 0. \quad (6)$$

Since the wave function  $u(r)$  vanishes at the origin, our first boundary condition on  $P(r)$  reads

$$P(0) = 0. \quad (7)$$

From eq. (6) and the condition above it follows that:

$$P'(0) = 0. \quad (8)$$

Finally, the probability of finding the particle somewhere must be unity, hence we impose the condition

$$\int_0^\infty P(r)dr = 1. \quad (9)$$

## 4 Energy corrections

In the framework of the probability density equation given in (6) one cannot exploit the orthonormality of wave functions directly. We shall therefore adopt a similar procedure to that presented in [2].

Suppose that the probability density eq. (6) is soluble for some unperturbed reduced local and Kisslinger potentials  $U_0(r)$  and  $\rho_0(r)$  with  $P_0(r)$  as a solution. Then introduce local and Kisslinger perturbations  $\lambda U_1(r)$  and  $\lambda \rho_1(r)$  respectively, so that

$$U(r) = U_0(r) + \lambda U_1(r), \quad \rho(r) = \rho_0(r) + \lambda \rho_1(r), \quad (10)$$

where  $U(r)$  and  $\rho(r)$  are real functions of the radial variable  $r$ . Now expand the probability densities and the eigenenergies in power series in  $\lambda$

$$P(r) = P_0(r) + \lambda P_1(r) + \lambda^2 P_2(r) + \lambda^3 P_3(r) + \dots, \\ E = E_0 + \lambda E_1 + \lambda^2 E_2 + \lambda^3 E_3 + \dots \quad (11)$$

For bound states the probability density must be normalized to unity for all values of the expansion parameter  $\lambda$ . It then follows from (9) that:

$$\int_0^\infty P_n(r)dr = \delta_{n0}. \quad (12)$$

Further, as we are dealing with bound states, the individual  $P_n(r)$  terms must decrease exponentially at large  $r$ , while the behavior of each at the origin is governed by eqs. (7) and (8).

Upon substituting for  $P(r)$  and  $E$  in (6) using (11), the resulting differential equation is satisfied provided that the coefficient of each power of  $\lambda$  vanishes for all  $r$ . Setting the coefficient of  $\lambda^0$  to zero results in the unperturbed equation

$$(1-\rho_0) \left[ P_0 P_0'' - \frac{(P_0')^2}{2} \right] - 2P_0^2(U_0 - E_0) \\ - \rho_0' P_0 P_0' + 2 \frac{\rho_0' P_0^2}{r} = 0. \quad (13)$$

The dependence of the variables on  $r$  has been omitted for clarity of presentation.

#### 4.1 First-order energy and probability density corrections

The coefficient of  $\lambda^1$  reads

$$\begin{aligned} & -4P_0P_1(U_0 - E_0) - 2P_0^2(U_1 - E_1) \\ & + (1 - \rho_0) [P_0P_1'' + P_0''P_1 - P_0'P_1'] \\ & + \rho_0' \left[ \frac{4P_0P_1}{r} - P_0'P_1 - P_0P_1' \right] + \rho_1 \left[ \frac{P_0'^2}{2} - P_0P_0'' \right] \\ & + \rho_1' \left[ \frac{2P_0^2}{r} - P_0P_0' \right] = 0. \end{aligned} \quad (14)$$

Substituting for  $(U_0 - E_0)$  using eq. (13) and then dividing by  $P_0$ , we obtain

$$\begin{aligned} 2P_0(U_1 - E_1) &= (1 - \rho_0) \frac{d}{dr} \left[ P_1' - \frac{P_1P_0'}{P_0} \right] \\ &+ \rho_0' \left[ \frac{P_1P_0'}{P_0} - P_1' \right] + \rho_1 \left[ \frac{P_0'^2}{2P_0} - P_0'' \right] + \rho_1' \left[ \frac{2P_0}{r} - P_0' \right]. \end{aligned} \quad (15)$$

Integrating the above leads to

$$\begin{aligned} 2E_1 \int_0^\infty P_0 dr &= 2 \int_0^\infty U_1 P_0 dr \\ &- \left\{ (1 - \rho_0) \left[ P_1' - \frac{P_1P_0'}{P_0} \right] \right\}_0^\infty \\ &- \int_0^\infty \left[ \rho_1 \frac{P_0'^2}{2P_0} + \rho_1' \frac{2P_0}{r} \right] dr. \end{aligned} \quad (16)$$

In the neighborhood of the origin, the term  $P_0'/P_0$  behaves like  $1/r$ , while  $P_1$  vanishes at least as fast as  $r^2$ . On the other hand, as  $r \rightarrow \infty$ ,  $P_0'/P_0$  is finite and  $P_1 \rightarrow 0$ . Consequently, the second term on the right is zero. Using eq. (12) we express the first-order energy correction as

$$E_1 = \int_0^\infty U_1(r) P_0 dr - \int_0^\infty \left[ \rho_1(r) \frac{P_0'^2}{4P_0} + \rho_1'(r) \frac{P_0}{r} \right] dr. \quad (17)$$

Clearly, the first-order energy correction is given purely in terms of the unperturbed probability density, the perturbation in the local potential, the perturbation in the velocity-dependent potential and its gradient. The dependence on  $\rho'$  reflects the importance of a diffuse edge in nuclei. This is particularly important in light nuclei.

In order to obtain the first-order correction of the probability density one should carry out an indefinite integration of eq. (15) which leads to

$$\begin{aligned} \frac{d}{dr} \left( \frac{P_1}{P_0} \right) &= \frac{2}{(1 - \rho_0)P_0} \int_0^r (U_1 - E_1) P_0 dr' \\ &- \frac{1}{(1 - \rho_0)P_0} \int_0^r \left( \rho_1 \frac{P_0'^2}{2P_0} + \rho_1' \frac{2P_0}{r'} \right) dr'. \end{aligned} \quad (18)$$

Integrating with respect to  $r$  results in

$$\begin{aligned} P_1(r) &= P_0 \int_0^r \frac{dr'}{(1 - \rho_0(r))P_0} \int_0^{r'} \left[ 2(U_1(r) - E_1)P_0 \right. \\ &- \rho_1(r) \frac{P_0'^2}{2P_0} - \rho_1'(r) \frac{2P_0}{r''} \left. \right] dr'' \\ &- P_0 \int_0^\infty P_0 dr \int_0^r \frac{dr'}{(1 - \rho_0(r))P_0} \\ &\times \int_0^{r'} \left[ 2(U_1(r) - E_1)P_0 - \rho_1(r) \frac{P_0'^2}{2P_0} - \rho_1'(r) \frac{2P_0}{r''} \right] dr'', \end{aligned} \quad (19)$$

where the constant of integration, multiplying  $P_0$  in the second line of the last equation, has been obtained by demanding that the integral of  $P_1$  over all  $r$  must vanish as required by eq. (12).

#### 4.2 Second-order corrections

The coefficient of  $\lambda^2$  reads

$$\begin{aligned} & 2E_2P_0^2 - (4P_0P_2 + 2P_1^2)(U_0 - E_0) - 4P_0P_1(U_1 - E_1) \\ & + (1 - \rho_0) \left[ P_0P_2'' + P_0''P_2 + P_1P_1'' - P_0'P_2' - \frac{P_1'^2}{2} \right] \\ & - \rho_0' \left[ P_0P_2' + P_0'P_2 + P_1P_1' - \frac{2P_1^2}{r} - \frac{4P_0P_2}{r} \right] \\ & + \rho_1 [P_0'P_1' - P_0''P_1 - P_0P_1''] \\ & + \rho_1' \left[ \frac{4P_0P_1}{r} - P_0P_1' - P_0'P_1 \right] = 0. \end{aligned} \quad (20)$$

Substituting for  $(U_0 - E_0)$  and  $3(U_1 - E_1)$  using eqs. (13) and (14) respectively, and then dividing by  $P_0$ , one arrives at

$$\begin{aligned} 2E_2P_0 - P_1(U_1 - E_1) &= \\ & (1 - \rho_0) \frac{d}{dr} \left[ \frac{P_0'P_2}{P_0} + \frac{P_1'P_1}{2P_0} - \frac{P_0'P_1^2}{2P_0^2} - P_2' \right] \\ & - \rho_0' \left[ \frac{P_0'P_2}{P_0} + \frac{P_1'P_1}{2P_0} - \frac{P_0'P_1^2}{2P_0^2} - P_2' \right] \\ & + \rho_1 \left[ \frac{3P_0'^2P_1}{4P_0^2} - \frac{P_0'P_1'}{P_0} - \frac{P_0''P_1}{2P_0} + P_1'' \right] \\ & - \rho_1' \left[ \frac{P_0'P_1}{2P_0} + \frac{P_1}{r} - P_1' \right]. \end{aligned} \quad (21)$$

Integrating the above over all  $r$  and using the normalization condition (12) the second-order energy correction may be put in the form

$$\begin{aligned} E_2 &= \frac{1}{2} \int_0^\infty U_1(r) P_1 dr \\ &+ \frac{1}{2} \int_0^\infty \left[ \rho_1(r) \left( \frac{P_1P_0'^2}{4P_0^2} - \frac{P_1'P_0'}{2P_0} \right) - \rho_1'(r) \frac{P_1}{r} \right] dr. \end{aligned} \quad (22)$$

As before, the correction depends on the perturbations and the gradient of the Kisslinger perturbing potential. The second-order probability density correction may be determined by evaluating the indefinite integral of (21) and then integrating the resulting with respect to  $r$ . This leads to

$$P_2 = P_0 \int_0^r \frac{dr'}{(1-\rho_0)P_0} \int_0^{r'} \left[ P_1(U_1 - E_1) - 2E_2P_0 + \rho_1 \frac{P_0'^2 P_1}{4P_0^2} - \rho_1' \frac{P_1}{r} \right] dr'' + \frac{P_1^2}{4P_0^2} + CP_0, \quad (23)$$

The constant of integration,  $C$ , is found by demanding that the integral of  $P_2(r)$  over all  $r$  must vanish. The third-order energy correction may be obtained in the same manner, however the algebra is more involved.

All the perturbation results obtained so far are independent of the unperturbed local potential  $U_0(r)$ . Therefore, the corrections obtained are also valid for any angular-momentum quantum number  $l$ , as the centrifugal barrier may be regarded as a part of  $U_0(r)$ .

## 5 Soluble potentials

In this section we shall determine the exact energy eigenvalues for eq. (2) using soluble potentials and, for simplicity, we shall consider the  $s$ -wave case.

### 5.1 Finite square-well potential

Using this type of potential, the differential equation involved may be solved exactly allowing an accurate determination of the energy eigenvalues. Such energies are then compared with the corresponding ones obtained using the formalism of the perturbation theory derived above. Our ansatz for the potentials and the corresponding perturbations are as follows:

$$U(r) = U_0(r) + \lambda U_1(r), \quad \rho(r) = \rho_0(r) + \lambda \rho_1(r), \quad (24)$$

where

$$\begin{aligned} U_0(r) &= -U_0, & U_1(r) &= -U_1, & r &< a, \\ U_0(r) &= 0, & U_1(r) &= 0, & r &> a, \end{aligned} \quad (25)$$

$$\begin{aligned} \rho_0(r) &= \rho_0, & \rho_1(r) &= \rho_1, & r &< a, \\ \rho_0(r) &= 0, & \rho_1(r) &= 0, & r &> a, \end{aligned} \quad (26)$$

where  $a = 1$  is the common radius of both potentials. Obviously, the local and velocity-dependent parts as well as the corresponding perturbations are spherically symmetric functions of the radial variable  $r$ . It is worth noting that the wave function is continuous at  $r = a$ . However, there is a finite jump in the derivative of the wave function at the boundary due to the sharp edge of the Kisslinger potential. The behavior of the derivative of the wave function at the boundary can be determined by integrating the

Schrödinger equation across the boundary from  $a - \epsilon$  to  $a + \epsilon$  and then taking the limit as  $\epsilon \rightarrow 0$ . The resulting condition is

$$(1 - \rho_0)R'_<(r) = R'_>(r), \quad (27)$$

where  $R(r) = u(r)/r$  is the radial wave function and  $R'_<(r)$ ,  $R'_>(r)$  are the derivatives of the wave functions corresponding to  $r$  less and greater than  $a$ , respectively. Taking  $U_0 = 2.2$ ,  $\rho_0 = 0.5$  and  $U_1 = \rho_1 = 0$ , that is, the perturbation is switched off, it is only possible to sustain one bound state whose energy is  $-0.0103$  in units of  $\hbar^2/2m$ . Alternatively, one may solve eq. (6) directly inside and outside the well and then match the probability density solutions at the boundary.

Now let us introduce perturbations to the local and velocity-dependent potentials namely  $U_1 = 0.01$  and  $\rho_1 = 0.01$ , respectively. Using eq. (17) the first-order energy correction is  $E_1 = -0.0043$  in units of  $\hbar^2/2m$ . Further, according to eq. (22) the second-order correction is  $E_2 = -0.0005$ , which is only about 11% of the first-order correction. So up to and including second order in the perturbation the bound-state energy is  $E = -0.0151$ . The exact energy eigenvalue is obtained by solving the Schrödinger equation with  $U_0 = 2.21$  and  $\rho_0 = 0.51$  and  $U_1 = \rho_1 = 0$ . This gives  $-0.0144$  in good agreement with  $-0.0151$  obtained using the perturbation approach. The absolute percentage difference is only 5%.

At small perturbations the agreement is expected to be good. However, as the energy increases the discrepancy tends to grow larger. This can be seen by considering the following transformation on the wave function [11]:

$$u(r) = \frac{\chi(r)}{\sqrt{1-\rho(r)}}, \quad (28)$$

which transforms eq. (2) into

$$\chi''(r) + [E - U_e(r)]\chi(r) = 0, \quad (29)$$

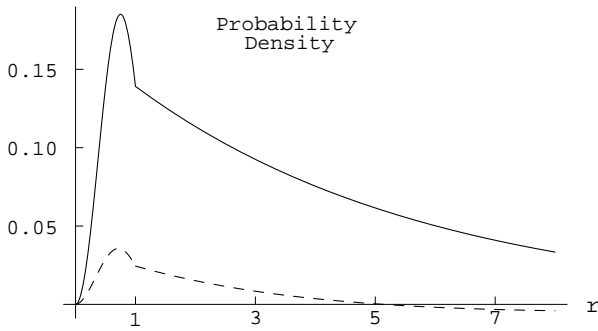
where  $U_e(r)$  is an effective potential that depends on the energy, *viz.*

$$U_e(r) = -\frac{1}{1-\rho(r)} \left[ \frac{\rho''(r)}{2} + \frac{(\rho'(r))^2}{4(1-\rho(r))} + \frac{\rho'(r)}{r} - U(r) + \rho(r)E \right]. \quad (30)$$

Obviously, the cross terms between  $U(r)$ ,  $E$  and  $\rho(r)$  can be ignored only when the perturbations are small. It is worth noting that the first- and second-order corrections in eqs. (17) and (22) are linear in  $U_1(r)$  and  $\rho_1(r)$  and contain no cross terms. Figure 1 shows the unperturbed probability density (solid line) and the corresponding first-order correction (dotted line), which becomes negative so that eq. (9) is satisfied. The discontinuity of the derivative at the boundary is clear.

## 6 Harmonic-oscillator potential

Here we shall take the local potential to be a finite square well as before, while the velocity-dependent part will be



**Fig. 1.** The unperturbed probability density (solid line) when the velocity-dependent term and the local potential are assumed to be finite square wells of the common radius  $a = 1$ . The corresponding first-order correction to the probability density (dotted line). Note that the first-order correction becomes negative. Further, the discontinuity of the derivative at  $r = 1$  is clear.

assumed to have a harmonic-oscillator behaviour, namely

$$\begin{aligned} \rho(r) &= \rho_0 r^2, & r < a, \\ &= 0, & r > a. \end{aligned} \quad (31)$$

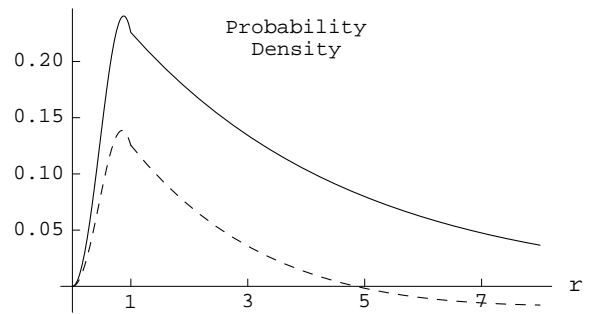
This potential demonstrates better the effect of the derivative of the potential than the finite square well considered earlier. Substituting for the potentials in (2) and solving the resulting differential equation results in the following solutions:

$$\begin{aligned} v(r) &= {}_2F_1(a, b, c, \eta), & r < a, \\ &= D e^{-\lambda r}, & r > a, \end{aligned} \quad (32)$$

where  ${}_2F_1(a, b, c, \eta)$  is the hypergeometric function, which upon evaluating for  $a, b, c$  and  $\eta$  becomes

$$\begin{aligned} u(r) &= Cr \left[ 1 - \frac{1}{6}(U_0 + E) r^2 \right. \\ &\quad + \frac{1}{120}(U_0 + E)(U_0 + E - 10\rho_0) r^4 \\ &\quad + \frac{1}{5040}(U_0 + E)(U_0 + E - 10\rho_0) \\ &\quad \left. \times (U_0 + E - 28\rho_0) r^6 + \dots \right], \end{aligned} \quad (33)$$

where  $C$  and  $D$  are constants that are determined using the appropriate boundary conditions at the boundary. The solution for  $r < a$  is a series with an infinite number of terms unless we terminate the series by setting one of the factors to zero. For example, let  $E = 10\rho_0 - U_0$ . For  $\rho_0 = 0.26$  the exact energy eigenvalue is  $-0.0174$ , in units of  $\hbar^2/2m$ , corresponding to  $U_0 = 2.62$ . Upon the introduction of the perturbing potentials  $\rho_1(r) = \rho_1 r^2 = 0.01 r^2$  and  $U_1 = 0.12$ , then using eq. (17), the first-order energy correction is  $-0.0188$ , while eq. (22) yields for the second-order correction a value of  $-0.0055$ . Thus, the eigenenergy up to and including the second-order correction is  $-0.0417$



**Fig. 2.** The unperturbed probability density (solid line) when the velocity-dependent term takes the form of a harmonic oscillator, while the local potential is assumed to be a finite square well. The corresponding first-order correction to the probability density (dotted line). Note that the first-order correction becomes negative. Further, the discontinuity of the derivative at  $r = 1$  is clear.

in units of  $\hbar^2/2m$ . However, an exact energy eigenvalue of  $-0.0379$  is obtained corresponding to  $\rho_0 = 0.27$  and  $U_0 = 2.74$ . The two values are in good agreement. The absolute percentage difference is about 10%. Figure 2 shows the unperturbed probability density (solid line) and the corresponding first-order correction (dotted line), which becomes negative so that the boundary condition in (9) is satisfied. Further, the discontinuity at the boundary, due to the discontinuity in the velocity-dependent part, is evident.

## 7 Discussion and conclusions

In this paper we have considered the partial-wave, time-independent Schrödinger equation in the presence of a velocity-dependent potential, which gives rise to a non-constant term by the second-order derivative. For stationary problems, where the potential is real, this equation has been written as an equation for the probability density. We have developed expressions for the first- and second-order energy shifts when spherically symmetric perturbations are added to the local and Kisslinger potentials. The derived expressions for the eigenenergy corrections are given in terms of the unperturbed probability density, the perturbation in the local potential, the Kisslinger perturbing potential and its gradient. No direct information regarding the unperturbed potentials are required. Further, the eigenfunctions for other states are not employed. The dependence on the gradient of the velocity-dependent potential stresses the importance of a diffuse edge in nuclei.

In order to test the validity of the derived expressions, two explicit examples were considered. In the first, each of the local and Kisslinger potentials was taken to be a finite square well of radius  $a = 1$ . Such forms of the potentials allow an exact determination of the energy eigenvalues. Upon the introduction of small perturbations in the potentials, perturbation theory formulas were used to calculate the energy eigenvalues up to and including the second-order correction. For small perturbations the agreement

between the exact and approximate values is found to be very satisfactory.

The energy corrections in eqs. (17) and (22) are linear in the potential perturbations  $\rho_1(r)$  and  $U_1(r)$ . However, by examining the transformed equation (29), where the potential becomes energy dependent, it can be seen that there are cross terms in  $\rho(r)$  with  $E$  and  $U(r)$ . Such terms are negligible when the perturbations in the potential are small. Consequently, the agreement between the exact eigenvalues and the approximate ones is better at small perturbing potentials and low energies.

In the second example the velocity-dependent potential is taken to have the form of a harmonic oscillator, namely  $\rho_1(r) = \rho_1 r^2$ . This demonstrates better the effect of the derivative of  $\rho(r)$  which is important in nuclear physics as nuclei, especially the light ones, have a diffuse edge. The Schrödinger equation was solved exactly and the results were in good agreement with the values obtained by the perturbation theory approach.

All the corrections obtained for the eigenenergies and probability densities are independent of the unperturbed local potential  $U_0(r)$ . This suggests that the corrections are valid in the presence of a centrifugal barrier term which may be contained in  $U_0(r)$ .

The Schrödinger equation with a non-constant term by the second-order derivative has been the interest of many research fields in recent years. In addition to nuclear physics, where the non-constant term may arise from a

spatially varying nuclear density, such an equation has been used to describe a particle with a spatially-dependent effective mass. The derived perturbation results may also be applied in such models.

The author would like to thank Prof. C. Wilkin for reading the manuscript and making helpful comments on it.

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